

Criteria for (in)finite extent of static perfect fluids

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Abstract

In Newton's and in Einstein's theory we give criteria on the equation of state of a barotropic perfect fluid which guarantee that the corresponding one-parameter family of static, spherically symmetric solutions has finite extent. These criteria are closely related to ones which are known to ensure finite or infinite extent of the fluid region if the assumption of spherical symmetry is replaced by certain asymptotic falloff conditions on the solutions. We improve this result by relaxing the asymptotic assumptions. Our conditions on the equation of state are also related to (but less restrictive than) ones under which it has been shown in Relativity that static, asymptotically flat fluid solutions are spherically symmetric. We present all these results in a unified way.

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1 Introduction

In this work we consider static, self-gravitating perfect fluids with a barotropic equation of state (EOS) $\rho(p)$ relating density and pressure, in Newton's theory (NT) and in General relativity (GR). We are thus dealing with time-independent, globally regular solutions to the Euler-Poisson and the Euler-Einstein equations. Some problems which have been studied in this context are the following:

- A. Do there exist (spherically symmetric, topologically $R^3 \times R$, asymptotically flat) solutions with a given equation of state ?
- B. Are all solutions (in particular all asymptotically flat ones) necessarily topologically $R^3 \times R$ and spherically symmetric ?
- C. Do fluid solutions (with parameters in some given range) have finite (or infinite) extent and a finite (or infinite) mass ?
- D. Can solutions (if they exist) be (uniquely) determined by parameters like surface potential, or mass (or central pressure or -density in the spherically symmetric case)?
- E. Are fluid solutions (with parameters in some given range) stable or unstable (against a certain classes of perturbations)?

Some of the questions listed above are clearly physically relevant when one develops models for stars or a model for the distribution of stars in a galaxy, while others are at least of mathematical interest. A recent review dealing with some of these issues can be found in [2]. We will (after making a few general remarks), focus here on the question of finiteness (point C.), summarizing previous work as well as presenting new results.

For smooth equations of state, existence of spherically symmetric solutions (point A) has been proven satisfactorily in NT and in GR [18], while for discontinuous equations of state an existence theory has been developed in NT only [19]. Regarding topology, partial results (in relativity) are given in [14]. As to spherical symmetry, a complete proof is available in NT [10], but the problem is not really settled in GR (see [12], and below). Conjectures on points A and B under rather general conditions will be formulated in the next section. In any case, it is useful to pose questions C, D and E in a general context, in particular without assuming spherical symmetry, since the latter is not required in some theorems. We also note that the available results on spherical symmetry in GR not only have to be formulated somewhat differently for infinite and finite configurations, but the proof of the finite case actually goes the detour via a uniqueness theorem for given surface potential (point D). Thus, in GR points A and B have so far been studied jointly with (parts of) C and D.

Turning now to the details of the finiteness problem, an interesting aspect is the relevance of the behaviour of $\rho(p)$ for small p . In fact, by modifying the EOS suitably in just an arbitrarily small neighbourhood of $p = 0$, all corresponding static solutions become necessarily finite or infinite. To see this, recall that fluid balls with $\rho(0) \neq 0$ are always finite [18], so we can simply set $\rho = \rho(\epsilon) = \text{const.}$ for $p \in [0, \epsilon]$ to ensure finiteness. On the other hand, (as we shall see below) we can always produce infinite solutions e.g. by gluing a piece with a linear EOS $p = C \cdot \rho$ for $p \in [0, \epsilon]$. We also recall that there are nontrivial criteria sufficient for finiteness of static fluid configurations, for example (assuming spherical symmetry) the existence of the integral $\int_0^p dp'/\rho(p')^2$ for some $p > 0$ (due to Rendall and Schmidt [18]) or $(\rho/p)(dp/d\rho) = \gamma + O(\rho^{\gamma-1})$ for a constant γ with $4/3 < \gamma < 2$ (due to Makino [13]), which only involve the behaviour of the EOS at small pressures.

These facts notwithstanding, it is important to realize that for the finiteness question the behaviour of $\rho(p)$ *for all* p is relevant in general. The striking examples in this context are the polytrope of index 5 in NT and the "Buchdahl" EOS in GR [6], viz.

$$p = \frac{1}{6}\rho_-^{-\frac{1}{5}}\rho^{\frac{6}{5}} \qquad p = \frac{1}{6}(\rho_-^{\frac{1}{5}} - \rho^{\frac{1}{5}})^{-1}\rho^{\frac{6}{5}} \quad (1)$$

where ρ_- is a positive constant, and $\rho < \rho_-$ in GR. All static solutions with these EOS have infinite extent but finite mass [4]. Criteria for (in)finiteness [22] (discussed and generalized below) imply, in particular, that these solutions are "on the verge" of having (in)finite extent and (in)finite mass in the following sense: Suitable, but arbitrarily small modifications of the EOS performed in an arbitrarily small neighbourhood of an arbitrary value p_0 of p , necessarily force *all* solutions with $p_0 \in \text{supp } \rho$ to be finite, while other small modifications near any p_0 force *all* solutions with $p_0 \in \text{supp } \rho$ to have infinite mass.

In this paper we discuss a theorem consisting of several parts. Under suitable conditions on the equation of state in each case, the theorem shows (roughly speaking) spherical symmetry from asymptotic flatness, finite and infinite extent of asymptotically flat fluid solutions, and finite extent in the spherically symmetric case. While the first part of the theorem is, in essence, just reproduced from [3, 11] and the second part is a technically improved version of [12, 22], the final part is the main new input of the present paper. The purpose of combining these three parts here is to exhibit the close relationship between the respective conditions on the equation of state (and in particular the role of (1) as a limiting case). As a complement to the present work, Heinzle has recently obtained results on finiteness and infiniteness of static fluids [8], assuming both asymptotic flatness and spherical symmetry, but less restrictive conditions on the equation of state than we do.

This report is organized as follows. In the next section we will give some preliminary material and quote the theorem discussed above. Crucial tools for the proof of the results on (in)finiteness are Pohozaev(-type) identities leading to virial(-type)

theorems, which we discuss in Sect.3. While in NT the virial theorem is an equality (without further assumptions), in GR one obtains in the same manner only a "virial inequality", (but equalities can still be derived when spherical symmetry is assumed [8]). Correspondingly, the "general" Newtonian virial theorem yields criteria both for finiteness and infiniteness of the fluid configuration, whereas in GR one only obtains a finiteness criterium in the same fashion, (but again an infiniteness criterium in the spherically symmetric case [8]). However, even without this latter assumption a criterium for infiniteness also exists in GR and is proven using the positive mass theorem with respect to a suitably conformally rescaled metric (a procedure which is, on the other hand, meaningless in NT). The proofs of the new parts of the main theorem are given in Sect.4. In the final section we expose our results in the light of the "quasipolytropic" family of EOS.

2 The main theorem

We will treat NT and GR in a "parallel" fashion as far as possible. This means that, if equations in NT and GR are direct analogues, we give them in left and right columns, respectively, and use the same symbols and the same numbers for corresponding quantities to facilitate comparison. In either situation, we consider a Riemannian manifold (\mathcal{M}, g) with a scalar function V . In the Conjecture and the Theorem below, we will take V and g to be in some Sobolev space L^q_{k+2} (using this symbol we follow Lee and Parker [9]), with exponent $q \geq 4$ and with k , the degree of (weak) differentiability, equal to either 0, 1 or 2. Furthermore, we consider on (\mathcal{M}, g) the non-negative functions $p \in L^q_{k+1}$, and $\rho \in L^q_k$, related by an EOS $\rho(p)$ which is C^{k-1} (if $k \geq 1$) and piecewise C^k . This means that for $k = 0$ we allow the density to "jump", while $k \geq 1$ forbids such jumps. We also require that (g, V) satisfy on \mathcal{M} (at least in a weak sense) the Euler-Poisson resp. the Euler-Einstein systems which we write as follows

$$\Delta V^{-1} = -4\pi\rho \qquad \Delta V = 4\pi(\rho + 3p)V \qquad (2)$$

$$R_{ij} = 0 \qquad R_{ij} = V^{-1}D_i D_j V + 4\pi(\rho - p)g_{ij} \qquad (3)$$

$$D_i p = \rho D_i V^{-1} \qquad D_i p = -V^{-1}(\rho + p)D_i V. \qquad (4)$$

The reason why we use here V^{-1} on the l.h. side will become clear shortly. In the GR case, \mathcal{M} can be understood as a hypersurface orthogonal to a timelike Killing vector ξ , and V denotes the norm of ξ . The covariant derivative D_i , the Laplacian $\Delta = D_i D^i$ and the Ricci tensor R_{ij} refer to g . As to the Euler equation, there is a difference between NT and GR: While in the former case we require that $p \in L^q_1$ and postulate (4), the corresponding (weak) version in GR can be derived as follows.

Lemma 1. (C.f. Remark 1.8. of [14].) Let (\mathcal{M}, g, V) be a L^q_2 solution which

satisfies the GR-side of the field equations (2) and (3) weakly. Then $p \in L_1^q$, and Euler's equation (4) also holds weakly.

Proof. Since C_0^∞ functions lie dense in L_k^q we can approximate g by a sequence of such functions. For any test function $\phi \in C_0^\infty$ we have then the following weak form of the Bianchi identity

$$\int_{\mathcal{M}} D^i \phi^j (R_{ij} - \frac{1}{2} R g_{ij}) d\mathcal{V} = 0 \quad (5)$$

where $d\mathcal{V}$ is the volume element w.r. to g . Approximating now also V, p and ρ by C_0^∞ -functions we easily obtain the required result. \square

While we allow ρ to have zeros in general (in particular at $p = 0$) we always assume that the integrals

$$\Gamma = \int_0^p \frac{dp'}{\rho(p')} \quad \Gamma = \int_0^p \frac{dp'}{\rho(p') + p'} \quad (6)$$

exist for some interval $p \in [0, p_{max}]$. (The integral does not exist e.g. for linear equations of state $\rho = C.p$). Obviously Γ is a continuous function of p and also on \mathcal{M} . In NT, from Euler's equation, we have $D_i(V^{-1} - \Gamma) = 0$ and so the expression in parenthesis is a constant V_s^{-1} equal to V^{-1} at $p = 0$, whence V_s is called the surface potential. In GR, the same conclusions can be drawn from $D_i(Ve^\Gamma) = 0$. Integrating (4) we obtain

$$\Gamma = V^{-1} - V_s^{-1} \quad \Gamma = \ln V_s - \ln V. \quad (7)$$

V is related to the standard Newtonian potential U by $U = 1 - V^{-1}$. To formulate our theorems we now introduce some quantities defined from the EOS, and proceed with a lemma on their mutual relationship. If $\rho(p)$ is C^0 and piecewise C^1 we define

$$\kappa = \frac{d\rho}{dp} \quad \kappa = \frac{\rho + p}{\rho + 3p} \frac{d\rho}{dp} \quad (8)$$

and if $\rho(p)$ is C^{k-1} (for $k = 1, 2$) and piecewise C^k (for $k = 0, 1, 2$),

$$I_0 = \rho\Gamma - 6p \quad I_0 = \rho(e^\Gamma - 1) - 6p \quad (9)$$

$$I_1 = 6p\kappa - 5\rho \quad I_1 = 6\frac{\rho + 3p}{\rho} p\kappa - 5\rho \quad (10)$$

$$I_2 = 5\rho \frac{d\kappa}{dp} + \kappa^2 \quad I_2 = 5(\rho + p) \frac{d\kappa}{dp} + \kappa^2 + 10\kappa. \quad (11)$$

The static spherically symmetric solutions corresponding to $I_0 \equiv 0$ and $I_1 \equiv 0$ are known explicitly and read (c.f. [5] for NT and [4, 6] for GR),

$$1 - V^{-1} = -\frac{M}{\sqrt{\frac{4\pi}{3}\rho_- M^4 + r^2}} \quad 1 - V = \frac{M}{\sqrt{\frac{4\pi}{3}\rho_- M^4 + r^2 + \frac{M}{2}}}, \quad (12)$$

where M is the mass which can take any positive value in NT. In GR, M is bounded from *below* by $3M^{-2} < 16\pi\rho_-$, and the metric on \mathcal{M} is $g = [(2/(1+V))^4 \delta]$ where δ is the Kronecker symbol.

We continue with a result on the relation between the quantities I_k .

Proposition 1. Assume that $\rho > 0$ for $p > 0$, and that $\lim_{p \rightarrow 0} \rho^{-1}p$ exists. Let $\{I_k \geq 0\}$, $\{I_k \equiv 0\}$ and $\{I_k \leq 0\}$ denote the sets of equations of state which are C^{k-1} (for $k = 1, 2$), piecewise C^k for $k = 0, 1, 2$ and which satisfy, for all $p \in [0, p_{max}]$, $I_k \geq 0$, $I_k \equiv 0$, or $I_k \leq 0$, respectively. Then

$$\{I_1 \geq 0\} \subseteq \{I_0 \geq 0\} \quad (13)$$

$$\{I_2 \equiv 0\} \supseteq \{I_1 \equiv 0\} \equiv \{I_0 \equiv 0\} \quad (14)$$

$$\{I_2 \leq 0\} \subseteq \{I_1 \leq 0\} \subseteq \{I_0 \leq 0\}. \quad (15)$$

Proof. These results are due to the fact that the I_j are in a certain sense the integrals of I_{j+1} . In particular, from (4), (7), (9) and (10) we obtain

$$\rho \frac{d}{dp} [\rho^{-1} I_0] = \rho^{-1} I_1 \quad (\rho + p) e^\Gamma \frac{d}{dp} [\rho^{-1} e^{-\Gamma} I_0] = \rho^{-1} I_1. \quad (16)$$

To show (13) and the second inclusion in (15) we note that $\rho^{-1} I_0$ vanishes at $p = 0$, which is obvious when $\rho_s \neq 0$. As to the case $\rho_s = 0$, we first note that the existence of Γ and of $\lim_{p \rightarrow 0} \rho^{-1}p$ implies $\lim_{p \rightarrow 0} \rho^{-1}p = 0$ (c.f. [8], Appendix A) and so the assertion follows from (9). Therefore, when $I_1 \geq 0$, $I_1 = 0$ or $I_1 \leq 0$, the same holds for I_0 .

Each of the conditions $I_1 \equiv 0$ and $I_0 \equiv 0$ (together with the differentiability assumptions) characterizes the 1-parameter families of EOS (1) and so the sets $\{I_1 \equiv 0\} \equiv \{I_0 \equiv 0\}$ of corresponding solutions are given by (12). The condition $I_2 = 0$ admits the case $\rho = \text{const.}$ and a two-parameter family of EOS which reads $p = 1/6(\rho_-^{-1/5} \rho^{6/5} - \rho_+)$ in NT (where ρ_- and ρ_+ are constants with $\rho_- > \rho_+$), while in GR it is given in [21] (together with the corresponding static solutions).

Finally, the first inclusion in (15) is proven in [3] (in a similar way as above). \square

We now impose falloff conditions on g and V in terms of weighted Sobolev spaces $L_{k,\beta}^q$ (defined as in Def. (2.1) of Bartnik [1], and Sect. 9 of Lee and Parker, [9] fol-

lowing again the notation of the latter paper).

Definition 1. A L_2^q solution (\mathcal{M}, g, V) of (2), (3) is called asymptotically flat (AF) if

1. The "end" $\mathcal{M}^\infty = \mathcal{M} \setminus \{\text{a compact set}\}$ is diffeomorphic to $R^3 \setminus \mathcal{B}$ where \mathcal{B} is a closed ball.
2. On \mathcal{M}^∞ , $1 - V = o(1)$, and the metric g satisfies, for some $q \geq 4$ and $\alpha > 0$,

$$g - \delta \in L_{2,\alpha}^q. \quad (17)$$

Thus, in this definition of AF we have supplemented Bartnik's falloff condition [1] for g by the requirement that $V \rightarrow \text{const.}$ at infinity (with the convention that $\text{const.} = 1$).

Definition 2. An AF solution (\mathcal{M}, g, V) of (2), (3) is called asymptotically flat with mass decay conditions (AFMD) if on \mathcal{M}^∞ the fluid variables satisfy, for some $q \geq 4$ and $\alpha > 0$,

$$\rho \in L_{0,-3-\alpha}^q, \quad (18)$$

$$p \in L_{0,-3-\alpha}^q. \quad (19)$$

Here the name "mass decay conditions" for these falloff conditions is inspired by similar requirements on the Ricci scalar in eqns.(4.4) of [1].

We remark that, to obtain our results in NT, it would be sufficient to impose (18) and (19) with "3" replaced by "5/2", which would not a priori restrict us to finite mass. In order not to spoil the "parallel" presentation of NT and GR, we just restrict ourselves to remarks on this option (after the formulation of the theorem).

Together with the field equations (2), (3), the AFMD conditions lead to the following stronger falloff properties. (We only give the formulation in GR explicitly; the Newtonian case arises via obvious simplifications.)

Lemma 2. Let (\mathcal{M}, g, V) be an AFMD GR solution. Then there exist a harmonic coordinate chart (\mathcal{M}^∞, g) and a constant M ("the mass") such that

$$1 - V - \frac{M}{r} \in L_{2,-1-\alpha}^q \quad (20)$$

$$g - (1 + 2M/r)\delta \in L_{2,-1-\alpha}^q \quad (21)$$

$$p \in L_{1,-4-\alpha}^q \quad (22)$$

Proof. Inserting $1 - V = o(1)$, (18) and (19) in (2) we have $\Delta V \in L_{0,-3-\alpha}^q$. Inverting

the Laplacian with Proposition 2.2 of [1] we obtain (20). The next step is to insert (18), (19) and (20) in (3) which yields $R_{ij} \in L_{0,-3-\alpha}^q$. By Proposition 3.3 of [1] we can now pass to harmonic chart on \mathcal{M}^∞ and get (21). Finally, (22) is obvious after inserting (20) and (21) in (4). \square

If we replace "3" by "5/2" in the definition of AFMD, we get Lemma 2 without the mass term, and with falloff $L_{2,-1/2-\alpha}^q$ for V, g and $L_{1,-3-\alpha}^q$ for p .

We now formulate a conjecture on points A,B and C raised in the introduction, and compare it with our theorem.

Conjecture. Let $\rho(p)$ be a piecewise C^0 equation of state with $\rho \geq 0$, $p \geq 0$. Then

- A:** There exists a 1-parameter family of spherically symmetric, AF L_2^q solutions (\mathcal{M}, g, V) of (2,3,4).
- B:** Every L_2^q -solution (\mathcal{M}, g, V) is necessarily spherically symmetric.
- C: I.** If $I_0 \leq 0$ holds for $p \in [0, p_{max}]$, then the solution either has finite extent or $\rho(p)$ satisfies $I_0 \equiv 0$.
- II.** If $I_0 \geq 0$ holds for $p \in [0, p_{max}]$, then either $\rho(p)$ satisfies $I_0 \equiv 0$ or the solution is not AF.

As remarked in Sect. 1, part A was shown under slightly stronger assumptions in NT [19] and for smooth EOS in GR [18]. We refer to these papers for details. As to spherical symmetry (B), the following theorem requires AFMD as well as a condition on the EOS, while the results on finiteness require either AFMD or spherical symmetry, together with stronger assumptions on the EOS than conjectured above.

Theorem. Let $\rho(p)$ be given with $p \geq 0$ and $\rho \geq 0$ and assume that there is a solution (\mathcal{M}, g, V) of (2,3,4) specified below.

- B:** Assume that $\rho(p)$ is C^1 with $d\rho/dp > 0$ and piecewise C^2 with $I_2 \leq 0$ for all $p \in [0, p_{max}]$, and assume that the solution is L_4^q ($q \geq 4$) and AFMD.

Then the solution is spherically symmetric.

- C: I.** If one of the following conditions 1. or 2. holds

1. $\rho(p)$ is piecewise C^0 , and $I_0 \leq 0$ holds for $p \in [0, p_{max}]$. The solution is AFMD.
2. $\rho(p)$ is C^0 , piecewise C^1 with $d\rho/dp > 0$, and $I_1 \leq 0$ holds for $p \in [0, p_{max}]$. The solution is L_3^q ($q \geq 4$) and spherically symmetric.

Then the solution either has finite extent, or $\rho(p)$ satisfies $I_0 \equiv 0$.

II. Assume that $\rho(p)$ is piecewise C^0 , and $I_0 \geq 0$ holds for all $p \in [0, p_{max}]$.

Then either $I_0 \equiv 0$, or the solution is not AFMD.

Some comments on this theorem are in order.

Whenever $I_0 \equiv 0$, then the EOS reads (1), and the solutions are given by (12).

Part B. of this theorem has been proven in [12] in the smooth case, but extends straightforwardly to the present setting.

Part C.I.1. has in essence been shown in [22], while C.II is proven in [22] and (assuming $I_1 \geq 0$) in [11]. Compared to these papers, we set out here from weaker assumptions on differentiability and on the falloff (as to NT, c.f. the remarks after Definition 2 and after Lemma 2).

The main new result of the present paper is part C.I.2. Its proof goes, in essence, the detour via C.I.1, i.e. we first show that assumptions C.I.2 (together with the field equations) imply falloff conditions similar to (20), (21) and (22) which can be used alternatively to the AFMD ones in C.I.1. in the spherically symmetric case. Since we also know from Proposition 1 that $I_1 \leq 0$ implies $I_0 \leq 0$, finite extent of the solution follows.

We note that C.I.1 and C.I.2 are also related in the sense that they require similar auxiliary results, (in particular Lemma 4) and so we found it useful to repeat (from [22]) the whole proof of C.I.1 here.

A natural extension of part C of the above theorem has been obtained by Heinzle [8]. From the equation of state he defines quantities J_{-1} in NT and J_0 in GR whose signs again determine the (in)finiteness properties of solutions which are now assumed to be both spherically symmetric and asymptotically flat. Moreover, as an extension of Proposition 1, it is shown in [8] that if I_0 has a sign (and if $\lim_{p \rightarrow 0} \rho^{-1}p$ exists), then J_{-1} and J_0 have the same sign as I_0 . Hence these quantities extend our " I_k series" in a natural manner, and Heinzle's (in)finiteness results require rather weak conditions on the equations of state.

3 The virial theorem

This section contains two easy technical lemmas followed by the "virial theorem" (as a proposition). The latter is a tool for the proof of the Theorem in Sect.4.

Lemma 3. (The modified Pohozaev identity [22]; compare [16]).

On (\mathcal{M}, g) with $\mathcal{M} \sim R^3$ and g flat, let ξ_i be a dilation, i.e. $D_{(i}\xi_{j)} = \delta_{ij}$ (in Cartesian coordinates x^i , $\xi_i = x^i$). Assume that $\sigma = \sigma(X)$ with $X \in L_2^q$, $\sigma \in L_0^q$ ($q \geq 4$) satisfies $\Delta X = 4\pi\sigma$, and define functions $\tau(X) = \int^X \sigma(X')dX'$ and $Z = D_j X D^j X$. (Here τ is defined only up to an additive constant). Then we have

$$D_i[(\xi^j D_j X + \frac{1}{2}X)D^i X - \frac{1}{2}Z\xi^i + 4\pi\tau\xi^i] = 2\pi(\sigma X + 6\tau). \quad (23)$$

This Lemma, which is proven by straightforward computation, will be applied below in NT, with $X = 1 - V^{-1}$, $\sigma = \rho$ and $\tau = p$.

In spaces in which there exists a (general) conformal Killing vector ξ , (i.e. $\mathcal{C}[D_i\xi_j] = 0$ where \mathcal{C} denotes the symmetric, trace-free part) there is an obvious generalization of (23). A more subtle generalization is the following.

Lemma 4. Let (\mathcal{M}, g') be a Riemannian 3-manifold with $g' \in L_2^q$ and Ricci-tensor and -scalar R'_{ij} , $R' = g'^{ij}R'_{ij}$ and $B'_{ij} = \mathcal{C}'[R'_{ij}]$ (where \mathcal{C}' denotes the symmetric, trace-free part w.r. to g'). Assume further that there exist a vector field $\eta^i \in L_1^q$ and a function $\alpha' \in L_0^q$ such that

$$\mathcal{C}'[D'_i\eta'_j] = \alpha' B'_{ij}. \quad (24)$$

Then

$$D'^i[(R'_{ij} - \frac{1}{2}g'_{ij}R')\eta'^j] = \alpha' B'_{ij}B'^{ij} - \frac{1}{6}R'D'_i\eta'^i \quad (25)$$

and if $g' \in L_3^q$,

$$D'^i[B'_{ij}\eta'^j] = \alpha' B'_{ij}B'^{ij} + \frac{1}{6}\eta'^i D'_i R'. \quad (26)$$

The derivatives on the l.h. sides on (25) and (26) are to be understood in the weak sense.

Proof. (25) and (26) follow from (24) by approximating η' with a sequence of C_0^∞ functions, and from the weak Bianchi identity (5) with respect to g' . \square

If there exists a conformal Killing vector ξ then we could in principle take $\eta = \xi$ in Lemma 4. Instead, we will make later use of this lemma with the more sophisticated choice

$$g' = (1 - V^{-1})^4 g \quad g' = \frac{1}{16}(1 + V)^4 g \quad (27)$$

$$\eta'_i = D_i(1 - V^{-1})^2 \quad \eta'_i = \frac{1 + V}{(1 - V)^3} D_i V \quad (28)$$

$$\alpha' = -(1 - V^{-1})^2 \quad \alpha' = \frac{V(1 + V)^2}{(1 - V)^4}. \quad (29)$$

As to the GR case of (25) we note that, using (27)-(29), the l.h. side can be rewritten (after multiplying with $(\det g'/\det g)^{1/2}$) as

$$D_i \left[\frac{1}{(1 - V^2)^2} \left(V^{-1} D^i W + 8 \frac{W D^i V}{1 - V^2} - 8\pi \left(\rho + \frac{1 - 5V}{1 - V} p \right) D^i V \right) \right], \quad (30)$$

where $W = D_i V D^i V$. The quantities on the r.h. side of (25) read explicitly

$$R' = 256\pi(1+V)^{-5}[\rho(1-V) - 6pV] \quad (31)$$

$$D'_i \eta'^i = 32(1-V^2)^{-4}[2\pi V(1-V^2)(\rho+3p) + 3W], \quad (32)$$

and so $D'_i \eta'^i \geq 0$.

We can now show the following

Proposition 2. (The "Virial theorem")

Let (\mathcal{M}, g, V) be AFMD. Then there is a $B \geq 0$ such that

$$\int_{\mathcal{M}} [\rho(1-V^{-1}) + 6p] d\mathcal{V} = 0 \quad \int_{\mathcal{M}} B[\rho(1-V^{-1}) + 6p] d\mathcal{V} \leq 0. \quad (33)$$

Proof. In NT we integrate the Pohozaev identity (23), apply the divergence theorem and note that the resulting surface terms vanish by virtue of the asymptotic properties derived in Lemma 2 (see also the remark after this lemma). In GR we apply the same reasoning to (25), using (31), $B = (128V\pi/3)(1+V)^{-5}D'_i \eta'^i$ and again Lemma 2 for the surface terms. \square

4 Proof of the main theorem

We first sketch the ideas of the proof. For part C.I.1 in general, and also for part C.II in NT we can use the virial theorem described above (which requires asymptotic flatness). We find that the sign of I_0 , together with $V_s < 1$ or $V_s = 1$, determine the signs of the r.h. sides of (33), which directly leads to the required conclusions. To show C.I.2 we introduce the quantities

$$\widehat{W} = (1-V^{-1})^{-4}W \quad \widehat{W} = (1-V^2)^{-4}W \quad (34)$$

$$\widehat{g} = (1-V^{-1})^4g \quad \widehat{g} = V^{-2}(1-V^2)^4g \quad (35)$$

$$\widehat{W}_0 = \frac{4\pi}{3} \int_{V_c}^V \frac{\rho V^4}{(1-V)^4} dV \quad \widehat{W}_0 = \frac{4\pi}{3} \int_{V_c}^V \frac{V(\rho+3p)}{(1-V^2)^4} dV, \quad (36)$$

where V_c is a constant specified below. We then employ (26), again with the choices (27), (28) and (29). As the r.h. side now contains $D'_i R'$ (instead of R' in (25)), controlling the sign now requires, (instead of $I_0 \leq 0$), the more restrictive condition $I_1 \leq 0$. In fact, assuming now also that the fluid region extends to infinity (i.e. $V_s = 1$) we can write (26) in the form

$$\widehat{\Delta}(\widehat{W} - \widehat{W}_0) \geq 0 \quad (37)$$

with $\widehat{W}_0 \geq C^2 = \text{const.} > 0$. If we were to assume that (\mathcal{M}, g, V) is AFMD, we could now form a compactification $\widehat{\mathcal{M}} = \mathcal{M} \cup \Lambda$ of this manifold by adding the point at infinity Λ , and \widehat{W} , \widehat{W}_0 and the metric \widehat{g} would have C^1 -extensions to Λ . Then the maximum principle would immediately lead again to the conclusions of C.I.1. Without asymptotic conditions we can still apply the maximum principle to (37) on a finite domain \mathcal{D} . Assuming also spherical symmetry and taking V_c in (36) to be the value of V at the centre, we conclude that $\widehat{W} \geq \widehat{W}_0 > C^2$, which gives $1 - V \leq 1/C.r$ on a ball \mathcal{B}_r of radius r . Thus, extending \mathcal{B}_r we control the asymptotic behaviour of V . With some further technical manipulations based on the fact that ΔV is positive, we can then prove that (\mathcal{M}, g, V) satisfies asymptotic conditions similar to the AFMD ones. These in fact guarantee the existence of a C^1 compactification as well as the vanishing of the surface terms in the integrals of (23) and (30). We then obtain the required conclusion $I_1 \equiv 0$ either from the maximum principle or by applying the reasoning of C.I.1.

We remark that, from (30), we could also write (26) in the form $\widehat{\Delta}(\widehat{W} - \widehat{W}_1) \geq 0$ for some $\widehat{W}_1(V)$ (assuming infinite extent and $I_1 \leq 0$). This relation can, however, not replace (37) in the proof sketched above as we do not have control over the sign of \widehat{W}_1 (whereas $\widehat{W}_0 > 0$).

Finally, to show part C.II in the Einstein case, we use the rigidity case of the positive mass theorem on (\mathcal{M}, g') .

We now turn to the details.

Proof of part C.I.1 of the Theorem. Using (7),(9) and the virial theorem (33), we find that

$$\int_{\mathcal{M}} [\rho(1 - V_s^{-1}) - I_0] d\mathcal{V} = 0 \qquad \int_{\mathcal{M}} B[(V_s - 1)(\rho + 6p) - I_0] d\mathcal{V} \leq 0, \quad (38)$$

where I_0 has here to be considered as a function on \mathcal{M} . The statement of the theorem now follows immediately from (38).

Proof of part C.II. In NT the second part of the theorem also follows easily from (38). As to the GR case, we find from (20) that the metric g' is $L_{2,\alpha}^q$, has vanishing mass, and $R' \geq 0$ due to (7), (9), (31) and $I_0 \geq 0$. The "vanishing mass-" case of the positive mass theorem [20] hence implies that g' is flat, and so R' vanishes identically. Using (31) together with (4) gives the Buchdahl EOS.

Proof of part C.I.2. We show that the requirements imply asymptotic conditions which in essence reduce the problem to C.I.1. We treat NT and GR simultaneously.

Observing from (31) that the Ricci scalar R' with respect to g' is a function of V , we first show that $I_1 \leq 0$ implies a lower bound for dR'/dV . To do so, we note first that the existence of Γ and $d\rho/dp > 0$ imply that $\lim_{p \rightarrow 0} \rho^{-1}p = 0$ (c.f. Appendix A

of [8]). By a straightforward computation, using $I_1 \leq 0$, $\rho^{-1}I_0 \leq 0$ (which follows from Proposition 1 and from the previous remark), and (7) we get

$$\begin{aligned}
\frac{(1-V)^6}{32\pi V^4 \rho \kappa} \frac{d}{dV} R' &= \frac{V(1+V)^6}{256\pi(\rho+3p)\kappa} \frac{d}{dV} R' = \\
&= 5\kappa^{-1} + 1 - V^{-1} \geq \\
&\geq \frac{6p}{\rho} + 1 - V^{-1} \geq \\
&\geq 1 - V^{-1} + \Gamma \\
&\geq 1 - V_s^{-1}
\end{aligned}
\qquad
\begin{aligned}
&= V^2(10\kappa^{-1} + 1) - 1 \geq \\
&\geq 12V^2 \left[3\frac{p^2}{\rho^2} + \frac{p}{\rho} + 1 \right] - 1 \geq \\
&\geq V^2 e^{2\Gamma} - 1 \\
&\geq V_s^2 - 1.
\end{aligned} \tag{39}$$

We now assume that the fluid extends to infinity, (i.e. $V_s = 1$), which will lead to a contradiction. It follows from (39) that $dR'/dV \geq 0$. Restricting ourselves to spherical symmetry we have $\widehat{W}_0 = 0$ at the centre by definition, and $\widehat{W} = 0$ since V takes its minimum there. Applying the maximum principle to (37) on any ball \mathcal{B}_r (bounded by $r = \text{const.}$) we obtain $\widehat{W} \geq \widehat{W}_0$ on \mathcal{B}_r . Next, since the integrand (36) is non-negative, $W_0(V)$ increases monotonically with V (and with r); in particular there is a constant $C > 0$ such that $\widehat{W} \geq \widehat{W}_0 \geq C^2$.

Below we give the proof of the relativistic case explicitly. (The Newtonian case can easily be obtained by suitable simplifications).

We introduce coordinates such that

$$ds^2 = g_{ij}dx^i dx^j = (1 - 2r^{-1}m(r))^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{40}$$

where $m(r) = 4\pi \int_0^r r'^2 \rho(r') dr'$. Writing out the bound $\widehat{W} \geq C^2$ explicitly we obtain, since $g^{rr} < 1$,

$$\frac{d}{dr} \frac{1}{1-V} = \frac{1}{(1-V)^2} \frac{dV}{dr} \geq \frac{\sqrt{g^{rr}}}{(1-V^2)^2} \frac{dV}{dr} = \sqrt{\widehat{W}} \geq C \tag{41}$$

which implies that

$$1 - V \leq \frac{1}{C.r}. \tag{42}$$

The crucial step is now to get an upper bound for $r^2 dV/dr$. Writing (2) as

$$\frac{d}{dr} \left(r^2 \sqrt{g^{rr}} \frac{dV}{dr} \right) = 4\pi V(\rho + 3p) r^2 \sqrt{g_{rr}} \tag{43}$$

and noting that the r.h. side is non-negative for $p > 0$, we find that dV/dr is positive and that $r^2 \sqrt{\widehat{W}}$ is strictly monotonically increasing. (Note that, from Theorem 2 of [18], g^{rr} does not go to zero for finite r . This argument also holds under the present

differentiability assumptions). We now show by an indirect argument that $r^2\sqrt{W}$ is bounded as $r \rightarrow \infty$. Assuming the contrary, namely that

$$\forall D > 0 \text{ and } \forall r_1 \exists r_0 > r_1 \text{ such that } r^2\sqrt{W}|_{r_0} > D \quad (44)$$

we get from monotonicity that

$$\forall D > 0 \exists r_0 \text{ such that } \forall r > r_0 \quad r^2\sqrt{W} > D \quad (45)$$

and, since $g^{rr} < 1$, we also have $r^2 dV/dr > D$ for sufficiently large r . Integrating the latter relation between some \hat{r} and infinity we get a contradiction to (42). Therefore, $r^2\sqrt{W}$ is bounded.

Using next the divergence theorem and $V_c \leq V$ (where V_c is the value of V at the centre) we obtain

$$\begin{aligned} m(r) &= \int_0^r \rho r'^2 dr' \leq \int_0^r \rho \sqrt{g_{rr}} r'^2 dr' \leq V_c^{-1} \int_0^r V(\rho + 3p) \sqrt{g_{rr}} r'^2 dr' = \\ &= V_c^{-1} \int_{r=\text{const.}} r^2 \sqrt{W} d\omega < \infty. \end{aligned} \quad (46)$$

Therefore, the ADM-mass $M = \lim_{r \rightarrow \infty} m(r)$ exists, $g^{rr} \rightarrow 1$, and the limit

$$\lim_{r \rightarrow \infty} r(1 - V) = \lim_{r \rightarrow \infty} r^2 \frac{dV}{dr} = \lim_{r \rightarrow \infty} r^2 \sqrt{W} = N \quad (47)$$

exists as well (and is called the "Komar mass"). Moreover, we note that $\rho \in L^1_{0,-3}$ and, from (4), that $p \in L^1_{1,-4}$. Finally, using (2) or (43) again, we obtain

$$\frac{dW}{dr} = -\frac{4}{r}W + 8\pi V(\rho + 3p)\sqrt{g_{rr}W} \in L^1_{0,-5}. \quad (48)$$

While these properties are slightly different from the AFMD conditions as defined in Sect.2., they still imply that the surface integral arising by applying the divergence theorem to (30) vanishes. To see this we rewrite the latter expression as follows (using spherical symmetry and multiplying with r^2),

$$\frac{d}{dr} \left[\frac{r^2}{(1 - V^2)^2} \left(V^{-1} \sqrt{g^{rr}} \frac{dW}{dr} + \frac{8W^{3/2}}{1 - V^2} - 8\pi \sqrt{W} \left(\rho - \frac{1 - 5V}{1 - V} p \right) \right) \right], \quad (49)$$

use the divergence theorem and insert (47), $r^4 W \rightarrow N^2$, and (48). Recalling that $I_1 \leq 0$ implies $I_0 \leq 0$, the virial theorem (33) now applies and we get finite extent.

As remarked before, it also follows that \hat{g} , \hat{W} and \hat{W}_0 have C^1 extensions to the point at infinity Λ of a compactification $\hat{\mathcal{M}} = \mathcal{M} \cup \Lambda$. Hence (instead of using the virial theorem in the form (33)) we could also apply the maximum principle to (37) to show finiteness.

□

5 Discussion

Here we discuss further the results on finiteness in the light of what we call the "quasipolytropic" EOS

$$p = K \rho^{\frac{n+1}{n}} \left[1 + f\left(\rho^{\frac{1}{n}}\right) \rho^{\frac{1}{n}} \right] \quad (50)$$

where $K, n \in \mathbb{R}^+$ and $f : [0, \lambda) \rightarrow \mathbb{R}$ is a smooth function. Obviously, for $f \equiv 0$, (but in general also for small ρ), we are left with the polytropic EOS, while the limit $n = 0$ reduces to the incompressible case $\rho = \text{const.}$. This quasipolytropic class seems to comprise all physically interesting EOS, in NT as well as in GR.

Let us first compare the polytropes in NT with the "Generalized Buchdahl EOS" in GR obtained from (50) with the choice $1/K = (n+1)\rho_-^{1/n}$ and $f(\rho^{1/n}) = (\rho_-^{1/n} - \rho^{1/n})^{-1}$ for some constant ρ_- , and for $\rho < \rho_-$, viz.

$$p = \frac{1}{n+1} \rho_-^{-\frac{1}{n}} \rho^{\frac{n+1}{n}} \quad p = \frac{1}{n+1} \rho^{\frac{n+1}{n}} (\rho_-^{\frac{1}{n}} - \rho^{\frac{1}{n}})^{-1}. \quad (51)$$

For the Newtonian polytropes we now find that $I_0 = (n-5)p$ and $I_1 = \rho(n-5)/(n+1)$. In GR, the integral (6) for Γ is still elementary but the expressions for I_0 and I_1 are rather involved. Nevertheless, they still have the appealing property that the quantities I_0 and I_1 are negative iff $n < 5$, zero for $n = 5$, (this is the "Buchdahl"-case considered above, c.f. (1),(12)) and positive iff $n > 5$. Hence the (in)finiteness properties of all static solutions (satisfying also the other requirements of C.I and C.II) depend only on the EOS and are obvious from the theorem.

While in NT the polytropes are also (for special values of n) adiabates and therefore physically significant, the adiabates in GR are neither polytropes nor do they coincide with (51). Instead, comparing again with the Newtonian polytropes, they read [7]

$$\rho = (n+1)^{\frac{n}{n+1}} \rho_-^{\frac{1}{n+1}} p^{\frac{n}{n+1}} \quad \rho = (n+1)^{\frac{n}{n+1}} \rho_-^{\frac{1}{n+1}} p^{\frac{n}{n+1}} + np. \quad (52)$$

The integral (6) for Γ is again elementary in the GR case of (52). We get $I_0 \leq 0$ (and hence finiteness for AFMD solutions) if

$$n \leq 5 \quad p \leq \frac{\rho_-}{n+1} \left(\frac{5}{n} - 1 \right)^{n+1}. \quad (53)$$

Thus adiabatic, asymptotically flat solutions whose pressure nowhere exceeds (53) are finite. The condition $I_1 \leq 0$, which guarantees finiteness for spherically symmetric solutions, leads to a (smaller) upper bound on p which can also be computed explicitly.

We now consider a fixed EOS in the general class (50) and the corresponding one-parameter family of static, spherically symmetric solutions. Rendall and Schmidt have shown [18] that, for $1 < n < 5$ the fluid region is finite provided that the

central density ρ_c is below a critical value ρ_{crit} , while for $n > 5$ the fluid is always infinite (with infinite mass). More recently, Makino has proven that the fluid is always finite for $1 < n < 3$. On the other hand, our theorem implies that finiteness for small ρ_c is guaranteed in the range $0 < n < 5$, and allows us to estimate ρ_{crit} . Now the interesting remaining question is what happens for $0 < n < 5$ if the central density is increased. For polytropes in GR, this question has been investigated by Nilsson and Ugglä [15] using a combination of dynamical systems techniques with numerical ones. They find that all solutions are finite for $n < 3.339$. For larger values they observe, somewhat surprisingly, a *discrete* set of central densities with infinite configurations, some of which have finite but others infinite mass. The mathematical side of these phenomena still deserves to be understood.

We finally remark that interesting results on finiteness have also been obtained for Vlasov-Poisson and Vlasov-Einstein systems by Rein and Rendall [17].

Acknowledgements. I am indebted to J. M. Heinzle for carefully reading the draft, pointing out errors and suggesting improvements. I am also grateful to R. Beig, P. Bizoň, A. Rendall, R. Steinbauer and C. Ugglä for helpful discussions and correspondence. This work was supported by Jubiläumsfonds der Österreichischen Nationalbank (Projekt 7942), and by Max-Planck-Institut für Gravitationsphysik.

References

- [1] Bartnik, R. (1986) The Mass of an Asymptotically Flat Manifold. *Comm. Pure Appl. Math.* 39, 661 - 693.
- [2] Beig, R. and Schmidt, B.G. (2000) Time-Independent Gravitational Fields", in: B.G. Schmidt (ed.) *Einstein's Field Equation and Their Physical Applications*. Lecture Notes in Physics, Springer, Berlin.
- [3] Beig, R., Simon, W. (1992) On the Uniqueness of Static Perfect-Fluid Solutions in General Relativity. *Commun. Math. Phys.* 144, 373 - 390.
- [4] Beig, R. and Simon, W. (1991) On the Spherical Symmetry of Static Perfect Fluids in General Relativity. *Lett. Math. Phys.* 21, 245 - 250.
- [5] Chandrasekhar, S. (1967) *An Introduction to the Study of Stellar Structure*. Dover Publications, New York.
- [6] Buchdahl, H. A. (1964) A Relativistic Fluid Sphere Resembling the Emden Polytrope of Index 5. *Astrophys. J.* 140, 1512 - 1516.
- [7] Demianski, M. (1985) *Relativistic Astrophysics* (Sect. 4.3). Pergamon Press, Oxford.

- [8] Heinzle, J. M. (2002) (In)finiteness of Spherically Symmetric Perfect Fluids. gr-qc/0204034.
- [9] Lee, J.M., Parker, T.H. (1987) The Yamabe Problem. Bull. Am. Math. Soc. 17, 37 - 91.
- [10] Lichtenstein, L. (1928) Über einige Eigenschaften der Gleichgewichtsfiguren rotierender Flüssigkeiten, deren Teilchen einander nach dem Newton'schen Gesetze anziehen. Math. Z. 28, 635 - 640.
- [11] Lindblom, L. and Masood-ul-Alam, A. K. M. (1993) Limits on the adiabatic index in static stellar models, in: B. L. Hu and T. Jacobson (eds.) Directions in general relativity. Cambridge University Press, Cambridge.
- [12] Lindblom, L. and Masood-ul-Alam, A. K. M. (1994) On the Spherical Symmetry of Static Stellar Models. Commun. Math. Phys. 162, 123 - 145.
- [13] Makino, T. (1998) On spherically symmetric stellar models in general relativity. Journ. Math. Kyoto Univ. 38, 55 - 69.
- [14] Masood-ul-Alam, A. K. M. (1987) The Topology of Asymptotically Euclidean Static Perfect Fluid Space-Time. Commun. Math. Phys. 108, 193 - 211.
- [15] Nilsson, U. S. and Uggla, C. (2000) General Relativistic Stars: Polytropic Equations of State. Ann. of Physics 286, 292 - 319.
- [16] Pohozaev, S. (1965) Eigenfunctions of the Equation $\Delta u + \lambda f(u) = 0$. Sov. Math. Dokl. 6, 1408 - 1411.
- [17] Rein, G. and Rendall, A. D. (2000) Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. Math. Proc. Cambridge Phil. Soc. 128, 363 - 380.
- [18] Rendall, A. D., Schmidt, B. G. (1991) Existence and properties of spherically symmetric static fluid bodies with a given equation of state. Class. Quantum Grav. 8, 985-1000.
- [19] Schaudt, U. (2000) On Static Stars in Newtonian Gravity and Lane-Emden Type Equations. Ann. Henri Poincaré 5, 945 - 976.
- [20] Schoen, R. and Yau, S-T. (1981) Proof of the positive mass theorem.II. Commun.Math.Phys. 79, 231 - 260.
- [21] Simon, W. (1994) A Class of Static Perfect Fluid Solutions. Gen. Rel. Grav. 26, 97 - 101.
- [22] Simon, W. (1993) On journeys to the Moon by balloon. Class. Quantum Grav. 10, 177 - 181.